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C-integrable nonlinear PDEs. IV

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Abstract. A technique based on a change of *independent* variables, used in a previous paper to generate C-integrable nonlinear PDEs (i.e. nonlinear PDEs solvable by an appropriate change of variables) in $1+1$ dimensions (one 'time' and one 'space' variable), is extended to the case of more than one space dimension. Several examples of evolution C-integrable PDEs in $1+2$ dimensions (one time and two space variables) are exhibited.

1. Introduction

This is the fourth of a series of papers devoted to C-integrable partial differential equations (PDEs), namely to nonlinear PDEs solvable via an appropriate change of variables. The first two papers [1, 2] of this series dealt with nonlinear evolution PDEs in $1+1$ dimensions (i.e. evolution PDEs involving one 'space' variable in addition to the 'time' variable) and focused, respectively, on equations solvable via a change of *dependent* and *independent* variables. The third paper [3] dealt with evolution equations involving more than only one space variable (in fact, it concentrated on the case of $1+2$ variables, i.e. one time variable and two space variables), solvable via an appropriate change of *dependent* variables. This fourth paper concentrates on evolution equations involving more than one space variable (in fact, again largely on the case of $1+2$ independent variables), solvable via an appropriate change of *independent* variables.

The motivation for deriving, and the criteria for exhibiting, such solvable nonlinear PDEs, have been outlined in the first two papers [1, 2] of this series.

2. A class of C-integrable equations in $1+N$ dimensions

Consider the *linear* evolution PDE

$$\boldsymbol{w}_\tau = \boldsymbol{B}\boldsymbol{w} + \sum_{k=1}^{N} \boldsymbol{A}_k \boldsymbol{w}_{\xi_k} . \tag{2.1}$$

Here the N 'space' variables ξ_k, together with the 'time' variable τ, are $N+1$ *independent* variables, the n-vector $\boldsymbol{w} \equiv \boldsymbol{w}(\tau, \xi_1, \xi_2, \ldots, \xi_N)$ is the *dependent* variable, and $\boldsymbol{A}_k, \boldsymbol{B}$ are $N+1$ constant $(n \times n)$-matrices. This PDE is of course solvable by standard techniques (for instance, within an appropriate functional class, by a Fourier transform of the space

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variables). The Cauchy (initial-value) problem identifies the solution characterized by the (initial) datum $\boldsymbol{w}(0, \xi_1, \xi_2, \ldots, \xi_N) = \boldsymbol{w}^{(0)}(\xi_1, \xi_2, \ldots, \xi_N)$.

We now introduce $N+1$ new independent variables x_j and t by setting

$$x_{j,\xi_k} = \lambda_{kj} + [H_j(\boldsymbol{w})]_{\xi_k} = \lambda_{kj} + (\nabla H_j, \boldsymbol{w}_{\xi_k}) \tag{2.2a}$$

$$x_{j,\tau} = \mu_j + [H_j(\boldsymbol{w})]_\tau = \mu_j \left(\nabla H_j, \boldsymbol{B}\boldsymbol{w} + \sum_{k=1}^{N} \boldsymbol{A}_k \boldsymbol{w}_{\xi_k} \right) \tag{2.2b}$$

$$t_{\xi_k} = 0 \qquad t_\tau = 1\,. \tag{2.2c}$$

Here of course $x_{j,\xi_k} \equiv \partial x_j(\tau, \xi_1, \xi_2, \ldots, \xi_N)/\partial \xi_k$, $x_{j,\tau} \equiv \partial x_j(\tau, \xi_1, \xi_2, \ldots, \xi_N)/\partial \tau$, and hereafter for simplicity we set $t = \tau$. The $N^2 + N$ quantities λ_{kj}, μ_j are real constants, and the N quantities $H_j(\boldsymbol{w})$ are (*a priori* arbitrary) real functions. The gradient ∇ in the right-hand side of (2.2*a*) and (2.2*b*) is the n-vector of components $(\partial/\partial w_1, \partial/\partial w_2, \ldots, \partial/\partial w_n)$, so that the p-component of the n-vector ∇H_j is $\partial H_j(\boldsymbol{w})/\partial w_p$. Note that to obtain the right-hand side of (2.2*b*) we have used (2.1), and that (2.2*a*) and (2.2*b*) automatically imply validity of the compatibility conditions $x_{j,\xi_k\xi_l} = x_{j,\xi_l\xi_k}$ and $x_{j,\tau\xi_k} = x_{j,\xi_k\tau}$.

We now enforce the *change of independent variables* (from the 'old' variables τ, ξ_k to the 'new' variables $t = \tau, x_j$ by setting

$$\boldsymbol{w}(\tau, \xi_1, \xi_2, \ldots, \xi_N) = \boldsymbol{u}(t, x_1, x_2, \ldots, x_N) \tag{2.3}$$

with (see equation (2.2))

$$x_j = \bar{x}_j + \sum_{k=1}^{N} \lambda_{kj}\xi_k + H_j[\boldsymbol{w}(t, \xi_1, \xi_2, \ldots, \xi_N)] \tag{2.4a}$$

$$x_j = \bar{x}_j + \sum_{k=1}^{N} \lambda_{kj}\xi_k + H_j[\boldsymbol{u}(t, x_1, x_2, \ldots, x_N)]\,. \tag{2.4b}$$

Here the quantities $\bar{x}_j$ are N (largely irrelevant, as long as we deal with autonomous equations) real constants, which are hereafter assumed to have been (conveniently) chosen, as well as the N^2 real constant λ_{kj} and the N real functions H_j. Equation (2.4*a*) provides an explicit definition of x_j in terms of the N coordinates ξ_k (whenever $\boldsymbol{w}(t, \xi_1, \xi_2, \ldots \xi_N)$ is known); it is the most convenient version to go from the old to the new variables. Equation (2.4*b*) provides an implicit definition of ξ_k in terms of the N coordinates x_j (whenever $\boldsymbol{u}(t, x_1, x_2, \ldots, x_N)$ is known); to get an explicit expression of ξ_k all that is needed is inversion of the constant $(N \times N)$-matrix $\boldsymbol{\Lambda}$ (of elements λ_{jk}), which is assumed to be invertible ($\det(\boldsymbol{\Lambda}) \neq 0$); this is the most convenient version to go from the new to the old variables. The fact that this change of independent variables is time dependent is shown by the explicit time dependence of the last term in the right-hand side of (2.4).

Let us now derive the nonlinear evolution equation satisfied by $\boldsymbol{u}(t, x_1, x_2, \ldots, x_N)$ (the reader who is not interested in the details of this derivation is advised to proceed directly to the result (2.14)).

From equations (2.3) and (2.2*a*) we get

$$\boldsymbol{w}_{\xi_k} = \sum_{j=1}^{N} \boldsymbol{u}_{x_j} x_{j,\xi_k} = \sum_{j=1}^{N} (\lambda_{kj} + H_{kj}^{(\xi)}) \boldsymbol{u}_{x_j} \tag{2.5}$$

where we have set

$$H_{kj}^{(\xi)} \equiv (\nabla H_j, \boldsymbol{w}_{\xi_k})\,. \tag{2.6a}$$

Let us also set

$$H_{kj}^{(x)} \equiv (\nabla H_j, \boldsymbol{u}_{x_k})\,. \tag{2.6b}$$

We thereby introduce the three $(N \times N)$-matrices $\boldsymbol{\Lambda}$, $\boldsymbol{H}^{(\xi)}$, $\boldsymbol{H}^{(x)}$ of elements λ_{jk}, $H_{jk}^{(\xi)}$, $H_{jk}^{(x)}$.

From the definitions (2.6) and from (2.5) we get the $(N \times N)$-matrix equation

$$\boldsymbol{H}^{(\xi)} = (\boldsymbol{\Lambda} + \boldsymbol{H}^{(\xi)})\boldsymbol{H}^{(x)} \tag{2.7}$$

which can be solved for $\boldsymbol{H}^{(\xi)}$:

$$\boldsymbol{H}^{(\xi)} = \boldsymbol{\Lambda}\boldsymbol{H}^{(x)}(\boldsymbol{I} - \boldsymbol{H}^{(x)})^{-1}\,. \tag{2.8a}$$

Here $\boldsymbol{I}$ is the unit $(N \times N)$-matrix.

Clearly equation (2.8*a*) entails

$$\boldsymbol{\Lambda} + \boldsymbol{H}^{(\xi)} = \boldsymbol{\Lambda}(\boldsymbol{I} - \boldsymbol{H}^{(x)})^{-1}\,. \tag{2.8b}$$

Let us set

$$(\boldsymbol{I} - \boldsymbol{H}^{(x)})^{-1} = \boldsymbol{Q}/\Delta \tag{2.9}$$

where

$$\Delta = \det(\boldsymbol{I} - \boldsymbol{H}^{(x)}) \tag{2.10}$$

and $\boldsymbol{Q}$ is the matrix adjoint to $\boldsymbol{I} - \boldsymbol{H}^{(x)}$.

It is now easy to compute $\boldsymbol{w}_{\xi_k}$ from equations (2.5) and (2.8*b*):

$$\boldsymbol{w}_{\xi_k} = \sum_{j,l=1}^{N} \lambda_{kj} Q_{jl} \boldsymbol{u}_{x_l}/\Delta\,. \tag{2.11}$$

Q_{jk} are the matrix elements of the $(N \times N)$-matrix $\boldsymbol{Q}$, see equations (2.9) and (2.10).

Likewise from (2.3),

$$\boldsymbol{w}_\tau = \boldsymbol{u}_t + \sum_{j=1}^{N} \boldsymbol{u}_{x_j} x_{j,\tau} \tag{2.12}$$

hence from (2.2*b*) and (2.3) we get

$$\boldsymbol{w}_\tau = \boldsymbol{u}_t + \sum_{j=1}^{N} \mu_j \boldsymbol{u}_{x_j} + \sum_{j=1}^{N} (\nabla H_j, \boldsymbol{B}\boldsymbol{u})\boldsymbol{u}_{x_j} + \sum_{j,k=1}^{N} (\nabla H_j, \boldsymbol{A}_k \boldsymbol{w}_{\xi_k})\boldsymbol{u}_{x_j} \tag{2.13}$$

(in the last term in the right-hand side of this equation the n-vector $\boldsymbol{w}_{\xi_k}$ should in fact be replaced by its expression (2.11); for notational simplicity we have refrained here from performing this substitution explicitly, which is however essential to get our final formula, see below).

Insertion of (2.13), (2.3) and (2.11) into (2.1) yields finally the C-integrable nonlinear evolution PDE satisfied by $\boldsymbol{u}(t, x_1, x_2, \ldots, x_n)$:

$$\boldsymbol{u}_t = \boldsymbol{B}\boldsymbol{U} - \sum_{j=1}^{N} \mu_j \boldsymbol{u}_{x_j} - \sum_{j=1}^{N} (\nabla H_j, \boldsymbol{B}\boldsymbol{u})\boldsymbol{u}_{x_j} + \sum_{j,k=1}^{N} Q_{kj} \left[\boldsymbol{M}_k \boldsymbol{u}_{x_j} - \sum_{m=1}^{N} (\nabla H_m, \boldsymbol{M}_k \boldsymbol{u}_{x_j}) \boldsymbol{u}_{x_m} \right] / \Delta \tag{2.14a}$$

with

$$\boldsymbol{M}_k = \sum_{j=1}^{N} \lambda_{jk} \boldsymbol{A}_j\,. \tag{2.14b}$$

Note that this is quite an explicit expression of the nonlinear PDE satisfied by $\boldsymbol{u}$. Of course, the round-bracketed expressions in the right-hand side of (2.14*a*) read explicitly as follows:

$$(\nabla H_j, \boldsymbol{B}\boldsymbol{u}) \equiv \sum_{a,b=1}^{n} [\partial H_j(\boldsymbol{u})/\partial_{u_a}](\boldsymbol{B})_{ab} u_b \tag{2.14c}$$

$$(\nabla H_m, \boldsymbol{M}_k \boldsymbol{u}_{x_j}) \equiv \sum_{a,b=1}^{n} [\partial H_m(\boldsymbol{u})/\partial_{u_a}](\boldsymbol{M}_k)_{ab} u_{b,x_j} \tag{2.14d}$$

where u_a is the a-component of the n-vector $\boldsymbol{u}$ and the matrix elements Q_{jk} of the $(N \times N)$-matrix $\boldsymbol{Q}$, as well as the determinant Δ, are defined by (2.9) and (2.10), with the matrix elements $H_{jk}^{(x)}$ of the $(N \times N)$-matrix $\boldsymbol{H}^{(x)}$ defined explicitly as follows (see equation (2.6*b*)):

$$H_{jk}^{(x)} = \sum_{a=1}^{n} [\partial H_k(\boldsymbol{u})/\partial u_a] u_{a,x_j} \, . \tag{2.15}$$

Much of the rest of this paper consists of more specific versions of this nonlinear PDE. But before exhibiting these examples, we indicate in the next subsection how to solve the Cauchy problem for (2.14).

2.1. Solution of the Cauchy problem

To understand the C-integrability of the nonlinear evolution PDE (equation (2.14)) we indicate here how to solve its Cauchy problem.

Let

$$\boldsymbol{u}(0, x_1, x_2, \ldots, x_N) = \boldsymbol{u}^{(0)}(x_1, x_2, \ldots, x_N) \tag{2.16}$$

be a *given* function of the N 'space' variables $x_1, x_2, \ldots, x_N$. Then by solving (2.4*b*) (with $t = 0$), one gets (at $t = 0$) the new variables x_j as a function of the old variables ξ_k (note that here, unfortunately, one has to use a less convenient version of the transformation from the old to the new variables, indeed one which generally requires the solution of a *nonlinear*—albeit *nondifferential*—equation).

Inserting the expression at $t = 0$ of the new variables x_j in terms of the old variables ξ_k into (2.3) (with $t = 0$) one gets the initial datum $\boldsymbol{w}(0, \xi_1, \xi_2, \ldots, \xi_N)$.

From this initial condition, one obtains $\boldsymbol{w}(t, \xi_1, \xi_2, \ldots, \xi_N)$ by solving the Cauchy problem for the *linear* evolution equation (1.1).

Insertion of $\boldsymbol{w}(t, \xi_1, \xi_2, \ldots, \xi_N)$ into (2.4*a*) gives a relation between new and old variables at time t, which must now be solved to get explicit expressions of the old variables ξ_k in terms of the new variables x_j (again here one is not using the most convenient version of the transformation, so that generally *nonlinear*—albeit *nondifferential*—equations have again to be solved).

Finally, using the expression of the old variables ξ_k in terms of the new variables x_j one gets, via (2.3), the solution $\boldsymbol{u}(t, x_1, x_2, \ldots, x_N)$ of the Cauchy problem for the evolution equation (2.14), corresponding to the initial datum (2.16).

3. Examples

In this section we exhibit several examples; we have tried to strike a reasonable balance between the possibility of presenting an infinite number of them, and the need to be as terse as possible, yet provide a representative sample. In the following, symbols such as a, b, c, β stand for (arbitrary) constants, $H_j(\boldsymbol{u})$ for (generally arbitrary) functions and $H_j' \equiv \partial H_j(u)/\partial u$.

Most of the following examples are explicit realizations of (2.14); others are obtained, by the same technique as described above, taking as a starting point other *linear* or (C-integrable) *nonlinear* PDEs.

3.1. $N = 2, n = 1$

Example 3.1.1.

$$u_t = \left[u_x + \frac{(u_x^2 + u_y^2)H_1'}{\Delta} \right]_x + \left[u_y + \frac{(u_x^2 + u_y^2)H_2'}{\Delta} \right]_y \tag{3.1a}$$

$$\Delta = 1 - H_1' u_x - H_2' u_y \,. \tag{3.1b}$$

It comes from the diffusion equation,

$$w_\tau = w_{\xi\xi} + w_{\eta\eta} \tag{3.2}$$

via the transformation (2.2) with

$$\lambda_{11} = \lambda_{22} = 1 \qquad \lambda_{12} = \lambda_{21} = 0 \qquad \mu_1 = \mu_2 = 0\,. \tag{3.3}$$

Of course here (and hereafter) $\xi_1 \equiv \xi, \xi_2 \equiv \eta, x_1 \equiv x, x_2 \equiv y$.

For instance, if $H_1(u) = au$, $H_2(u) = 0$, equation (3.1) reads as

$$u_t = u_{yy} + 2au_y u_{xy}/(1 - au_x) + (1 + a^2 u_y^2) u_{xx}/(1 - au_x)^2\,. \tag{3.4}$$

If $H_1(u) = \frac{1}{2}au^2$, $H_2(u) = 0$, equation (3.1) reads as

$$u_t = u_{yy} + 2auu_y u_{xy}/(1 - auu_x) \\ +[a(u_x^2 + u_y^2)u_x + (1 + a^2u^2u_y^2)u_{xx}]/(1 - auu_x)^2\,. \tag{3.5}$$

Example 3.1.2.

$$u_{tt} = -\beta^2 u + \left[u_x + \frac{(u_x^2 + u_y^2 - u_t^2)H_1'}{\Delta} \right]_x + \left[u_y + \frac{(u_x^2 + u_y^2 - u_t^2)H_2'}{\Delta} \right]_y\,. \tag{3.6}$$

Here Δ is given by (3.1*b*).

It comes from the Klein–Gordon equation:

$$w_{\tau\tau} = w_{\xi\xi} + w_{\eta\eta} - \beta^2 w \tag{3.7}$$

via the transformation (2.2) with (3.3).

For instance, if $H_1(u) = au$, $H_2(u) = 0$, equation (3.6) reads as

$$u_{tt} = -\beta^2 u + u_{yy} + \beta^2 auu_x + 2a(u_y u_{xy} - u_t u_{tx})/(1 - au_x) \\ +[1 + a^2(u_y^2 - u_t^2)u_{xx}/(1 - au_x)^2\,. \tag{3.8}$$

If $H_1(u) = \frac{1}{2}au^2$, $H_2(u) = 0$, equation (3.6) reads

$$u_{tt} = -\beta^2 u + u_{yy} + \beta^2 au^2 u_x + 2au(u_y u_{xy} - u_t u_{tx})/(1 - auu_x) \\ +\{a(u_x^2 + u_y^2 - u_t^2)u_x + [1 + a^2u^2(u_y^2 - u_t^2)]u_{xx}\}/(1 - auu_x)^2\,. \tag{3.9}$$

Example 3.1.3.

$$\mathrm{i}\psi_t + \left[\psi_x + \frac{(\psi_x^2+\psi_y^2)H_1'}{\Delta}\right]_x + \left[\psi_y + \frac{(\psi_x^2+\psi_y^2)H_2'}{\Delta}\right]_y = 0\,. \tag{3.10a}$$

Here Δ is given by (3.1*b*) with u replaced by ψ,

$$\Delta = 1 - H_1'\psi_x - H_2'\psi_y \tag{3.10b}$$

$$H_j' = \partial H_j(\psi)/\partial\psi\,. \tag{3.10c}$$

It comes from the linear Schrödinger equation.

$$\mathrm{i}\phi_\tau + \phi_{\xi\xi} + \phi_{\eta\eta} = 0 \tag{3.11}$$

via the transformation (2.2) with (3.3), with ϕ in place of w, and ψ in place of u.

Example 3.1.4.

$$\begin{aligned} u_t = a_{00}u - a_{00}(x-\bar{x}-H_1(u))u_x + a_{01}u_y \\ -a_{00}u(H_1'u_x + H_2'u_y) + a_{02}(u_x p_y - p_x u_y) \\ +(a_{20}u^2 + a_{11}pu + a_{02}p^2)[u_{xx} + (H_1'u_x^2/\Delta)_x + (H_2'u_x^2/\Delta)_y] \\ +(a_{11}u + 2a_{02}p)[u_{xy} + (H_1'u_xu_y/\Delta)_x + (H_2'u_xu_y/\Delta)_y] \\ +a_{02}[u_{yy} + (H_1'u_y^2/\Delta)_x + (H_2'u_y^2/\Delta)_y] \end{aligned} \tag{3.12a}$$

$$(1 - H_2'u_y)p_x + H_2'u_x p_y = u^{-1}(pu_x + u_y)\,. \tag{3.12b}$$

Here Δ is given by (3.1*b*). $\bar{x}$ is an arbitrary constant.

This pair of equations comes from the C-integrable PDE

$$w_\tau = a_{00}(w - \xi w_\xi) + a_{01}w_\eta + a_{20}w^2 w_{\xi\xi} + a_{11}w^2 v_{\xi\xi} + a_{02}(wv_{\xi\eta} + wvv_{\xi\xi}) \tag{3.13a}$$

$$v_\xi = (vw_\xi + w_\eta)/w \tag{3.13b}$$

(see [4] equation (3.162) with $a_{30} = a_{21} = a_{12} = a_{03} = 0$), via the transformation (2.2) with (3.3) and setting

$$w(\tau,\xi,\eta) = u(t,x,y) \qquad v(\tau,\xi,\eta) = p(t,x,y)\,. \tag{3.14}$$

3.2. $N = 2, n = 2$

Example 3.2.1.

$$\begin{aligned} u_{1t} = \mathrm{i}\beta u_1 - \mu_1 u_{1x} + (\lambda_{11} + \mathrm{i}\lambda_{21})u_{2x} - \mu_2 u_{1y} - \mathrm{i}\beta(au_1 - bu_2)u_{1x} \\ -b[(\lambda_{11} - \mathrm{i}\lambda_{21})u_{1x}^2 - (\lambda_{11} + \mathrm{i}\lambda_{21})u_{2x}^2]/(1 - au_{1x} - bu_{2x}) \end{aligned} \tag{3.15a}$$

$$\begin{aligned} u_{2t} = -\mathrm{i}\beta u_2 + (\lambda_{11} - \mathrm{i}\lambda_{21})u_{1x} - \mu_1 u_{2x} - \mu_2 u_{2y} - \mathrm{i}\beta(au_1 - bu_2)u_{2x} \\ +a[(\lambda_{11} - \mathrm{i}\lambda_{21})u_{1x}^2 - (\lambda_{11} + \mathrm{i}\lambda_{21})u_{2x}^2]/(1 - au_{1x} - bu_{2x})\,. \end{aligned} \tag{3.15b}$$

It comes from (2.14) with

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \mathrm{i}\beta & 0 \\ 0 & -\mathrm{i}\beta \end{pmatrix}. \tag{3.16a}$$

$$\lambda_{12} = \lambda_{22} = 0 \qquad \lambda_{11}, \lambda_{21}, \mu_1, \mu_2 \text{ arbitrary} \tag{3.16b}$$

$$H_1(\boldsymbol{u}) = au_1 + bu_2 \qquad H_2(\boldsymbol{u}) = 0\,. \tag{3.16c}$$

If $\mu_1, \mu_2, \lambda_{11}, \lambda_{21}$ are real constants and $b = a^*$, it is consistent to set

$$u_1(t,x,y) = \psi(t,x,y) \qquad u_2(t,x,y) = \psi^*(t,x,y)\,. \tag{3.17}$$

Then the evolution equation reads

$$\mathrm{i}\psi_t = -\beta\psi - \mathrm{i}\mu_1\psi_x + \mathrm{i}c\psi_x^* - \mathrm{i}\mu_2\psi_y + 2\mathrm{i}\beta\,\mathrm{Im}(a\psi)\psi_x + 2a^*\,\mathrm{Im}(c^*\psi_x^2)/[1 - 2\,\mathrm{Re}(a\psi_x)]\,. \tag{3.18}$$

Here $c = \lambda_{11} + \mathrm{i}\lambda_{21}$.

Example 3.2.2.

$$u_{1t} = \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} - \mathrm{i}\beta a u_1 u_{1x} - \mathrm{i}a(u_{1x}u_{2y} - u_{2x}u_{1y}) \tag{3.19a}$$

$$\begin{aligned} u_{2t} = {}& -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} - \mathrm{i}\beta a u_1 u_{2x} - \mathrm{i}a u_{2x}u_{2y} \\ & + a(u_{1x}^2 - u_{2x}^2 - \mathrm{i}u_{1x}u_{1y} - \mathrm{i}a u_{2x}^2 u_{1y})/(1 - a u_{1x})\,. \end{aligned} \tag{3.19}$$

It comes from (2.14) with (3.16a), (3.3b) and

$$H_1(\boldsymbol{u}) = a u_1 \qquad H_2(\boldsymbol{u}) = 0\,. \tag{3.20}$$

This pair of equations can be reduced (with $u = u_1$) to (3.8).

Example 3.2.3.

$$\begin{aligned} u_{1t} = {}& \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} - \mathrm{i}\beta(a u_1 - b u_2)u_{1x} + \beta(a u_1 + b u_2)u_{1y} \\ & - b[u_{1x}^2 - u_{2x}^2 - u_{1y}^2 - u_{2y}^2 - 2\mathrm{i}u_{1x}u_{1y} \\ & - 2a(\mathrm{i}u_{2x} - u_{2y})(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \end{aligned} \tag{3.21a}$$

$$\begin{aligned} u_{2t} = {}& -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} - \mathrm{i}\beta(a u_1 - b u_2)u_{2x} + \beta(a u_1 + b u_2)u_{2y} \\ & + a[u_{1x}^2 - u_{2x}^2 + u_{1y}^2 + u_{2y}^2 - 2\mathrm{i}u_{2x}u_{2y} \\ & + 2b(\mathrm{i}u_{1x} + u_{1y})(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \end{aligned} \tag{3.21b}$$

$$\Delta = 1 - a(u_{1x} + \mathrm{i}u_{1y}) - b(u_{2x} - \mathrm{i}u_{2y}) - 2\mathrm{i}ab(u_{1x}u_{2y} - u_{2x}u_{1y})\,. \tag{3.21c}$$

It comes from (2.14) with (3.16a), (3.3) and

$$H_1(\boldsymbol{u}) = a u_1 + b u_2 \qquad H_2(\boldsymbol{u}) = \mathrm{i}(a u_1 - b u_2)\,. \tag{3.22}$$

If $b = a^*$, it is consistent to set (3.17). Then the evolution equation reads

$$\begin{aligned} \mathrm{i}\psi_t = {}& -\beta\psi + \mathrm{i}\psi_x^* - \psi_y^* + 2\mathrm{i}\beta[\mathrm{Im}(a\psi)\psi_x + \mathrm{Re}(a\psi)\psi_y] \\ & + 2a^*[\mathrm{Im}(\psi_x^2) + \mathrm{i}\,\mathrm{Re}(\psi_y^2) - \psi_x\psi_y \\ & - 2a(\mathrm{i}\psi_x^* - \psi_y^*)\,\mathrm{Im}(\psi_x\psi_y^*)]/\Delta \end{aligned} \tag{3.23a}$$

$$\Delta = 1 - 2\,\mathrm{Re}(a\psi_x) + 2\,\mathrm{Im}(a\psi_y) + 4|a|^2\,\mathrm{Im}(\psi_x\psi_y^*)\,. \tag{3.23b}$$

Example 3.2.4.

$$u_{1t} = \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} + \mathrm{i}\beta(au_2u_{1x} - bu_1u_{1y}) + \{a[-u_{1x}^2 + u_{2x}^2 + \mathrm{i}(u_{1x}u_{1y} + u_{2x}u_{2y})] + [b + a(\mathrm{i}au_{1x} + bu_{2x}) - b(bu_{1y} - \mathrm{i}au_{2y})] \times (u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \tag{3.24a}$$

$$u_{2t} = -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} + \mathrm{i}\beta(au_2u_{2x} - bu_1u_{2y}) - \{b[-u_{1x}u_{1y} + u_{2x}u_{2y} + \mathrm{i}(u_{1y}^2 + u_{2y}^2)] + [\mathrm{i}a - a(bu_{1x} + \mathrm{i}au_{2x}) + b(\mathrm{i}au_{1y} + bu_{2y})] \times (u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \tag{3.24b}$$

$$\Delta = 1 - (au_{2x} + bu_{1y}) - ab(u_{1x}u_{2y} - u_{2x}u_{1y})\,. \tag{3.24c}$$

It comes from (2.14) with (3.16*a*), (3.3) and

$$H_1(\boldsymbol{u}) = au_2 \qquad H_2(\boldsymbol{u}) = bu_1\,. \tag{3.25}$$

Example 3.2.5.

$$u_{1t} = \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} - \mathrm{i}\beta(au_1 - bu_2)u_{1y} + \{b[-u_{1x}u_{1y} + u_{2x}u_{2y} + \mathrm{i}(u_{1y}^2 + u_{2y}^2)] + [a - (a^2 - b^2)u_{1y}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \tag{3.26a}$$

$$u_{2t} = -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} - \mathrm{i}\beta(au_1 - bu_2)u_{2y} - \{a[-u_{1x}u_{1y} + u_{2x}u_{2y} + \mathrm{i}(u_{1y}^2 + u_{2y}^2)] + [b - (b^2 - a^2)u_{2y}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \tag{3.26b}$$

$$\Delta = 1 - au_{1y} - bu_{2y}\,. \tag{3.26c}$$

It comes from (2.14) with (3.16*a*), (3.3) and

$$H_1(\boldsymbol{u}) = 0 \qquad H_2(\boldsymbol{u}) = au_1 + bu_2\,. \tag{3.27}$$

If $b = a^*$, it is consistent to set (3.17). Then the evolution equation reads

$$\mathrm{i}\psi_t = -\beta\psi + \mathrm{i}\psi_x^* - \psi_y^* + 2\mathrm{i}\beta\,\mathrm{Im}(a\psi)\psi_y - 2\{a^*[\mathrm{Re}(\psi_y^2) - \mathrm{Im}(\psi_x\psi_y)] + [a - 2\mathrm{i}\,\mathrm{Im}(a^2)\psi_y]\,\mathrm{Im}(\psi_x\psi_y^*)\}/[1 - 2\,\mathrm{Re}(a\psi_y)]\,. \tag{3.28}$$

Example 3.2.6.

$$u_{1t} = \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} - \mathrm{i}\beta a u_1^2(u_{1x} + \mathrm{i}u_{1y}) \tag{3.29a}$$

$$u_{2t} = -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} - \mathrm{i}\beta a u_1^2(u_{2x} + \mathrm{i}u_{2y}) + au_1(u_{1x}^2 - u_{2x}^2 + u_{1y}^2 + u_{2y}^2 - 2\mathrm{i}u_{2x}u_{2y})/[1 - au_1(u_{1x} + \mathrm{i}u_{1y})]\,. \tag{3.29b}$$

It comes from (2.14) with (3.16*a*), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}au_1^2 \qquad H_2(\boldsymbol{u}) = \tfrac{1}{2}\mathrm{i}au_1^2\,. \tag{3.30}$$

Example 3.2.7.

$$u_{1t} = \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} - \mathrm{i}\beta(au_1^2 - bu_2^2)u_{1x} + \{bu_2[(u_{2x}^2 - u_{1x}^2) + \mathrm{i}(u_{1x}u_{1y} + u_{2x}u_{2y})] - \mathrm{i}[au_1 - (a^2u_1^2 + b^2u_2^2)u_{1x}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \quad (3.31a)$$

$$u_{2t} = -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} - \mathrm{i}\beta(au_1^2 - bu_2^2)u_{2x} + \{au_1[(u_{1x}^2 - u_{2x}^2) - \mathrm{i}(u_{1x}u_{1y} + u_{2x}u_{2y})] - \mathrm{i}[bu_2 - (a^2u_1^2 + b^2u_2^2)u_{2x}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \quad (3.31b)$$

$$\Delta = 1 - \tfrac{1}{2}(au_1^2 + bu_2^2)_x\,. \quad (3.31c)$$

It comes from (2.14) with (3.16*a*), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}(au_1^2 + bu_2^2) \qquad H_2(\boldsymbol{u}) = 0\,. \quad (3.32)$$

If $b = a^*$, it is consistent to set (3.17). Then the evolution equation reads

$$\mathrm{i}\psi_t = -\beta\psi + \mathrm{i}\psi_x^* - \psi_y^* + 2\mathrm{i}\beta\,\mathrm{Im}(a\psi^2)\psi_x + 2\{-a^*\psi^*[\mathrm{Im}(\psi_x^{*2}) + \mathrm{Re}(\psi_x\psi_y)] + \mathrm{i}\,\mathrm{Im}(\psi_x\psi_y^*)[a\psi - 2\,\mathrm{Re}(a^2\psi^2)\psi_x]\}/[1 - 2\,\mathrm{Re}(a\psi\psi_x)]\,. \quad (3.33)$$

Example 3.2.8.

$$u_{1t} = \mathrm{i}\beta u_1 + u_{2x} + \mathrm{i}u_{2y} + \{u_1[b(u_{2x}^2 - u_{1x}^2) + \mathrm{i}a(u_{1y}^2 + u_{2y}^2) - (a - \mathrm{i}b)u_{1x}u_{1y} + (a + \mathrm{i}b)u_{2x}u_{2y}] + [(a - \mathrm{i}b)u_2 + [(a + \mathrm{i}b)u_1^2 - (a - \mathrm{i}b)u_2^2](bu_{1x} + au_{1y})] \times (u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \quad (3.34a)$$

$$u_{2t} = -\mathrm{i}\beta u_2 + u_{1x} - \mathrm{i}u_{1y} + \{u_2[b(u_{1x}^2 - u_{2x}^2) - \mathrm{i}a(u_{1y}^2 + u_{2y}^2) + (a - \mathrm{i}b)u_{1x}u_{1y} - (a + \mathrm{i}b)u_{2x}u_{2y}] + [-(a + \mathrm{i}b)u_1 + [(a + \mathrm{i}b)u_1^2 - (a - \mathrm{i}b)u_2^2](bu_{2x} + au_{2y})] \times (u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \quad (3.34b)$$

$$\Delta = 1 - b(u_1u_2)_x - a(u_1u_2)_y\,. \quad (3.34c)$$

It comes from (2.14) with (3.16*a*), (3.3) and

$$H_1(\boldsymbol{u}) = bu_1u_2 \qquad H_2(\boldsymbol{u}) = au_1u_2\,. \quad (3.35)$$

If a and b are real, it is consistent to set (3.17). Then the evolution equation reads as

$$\mathrm{i}\psi_t = -\beta\psi + \mathrm{i}\psi_x^* - \psi_y^* - 2\{\psi[b\,\mathrm{Im}(\psi_x^{*2}) + a\,\mathrm{Re}(\psi_y^{*2}) - \mathrm{Im}((a - \mathrm{i}b)\psi_x\psi_y)] + \mathrm{Im}(\psi_x\psi_y^*)[(a - \mathrm{i}b)\psi^* + 2\mathrm{i}(b\psi_x + a\psi_y)\,\mathrm{Im}((a + \mathrm{i}b)\psi^2)]\}/\Delta \quad (3.36a)$$

$$\Delta = 1 - 2[b\,\mathrm{Re}(\psi\psi_x^*) + a\,\mathrm{Re}(\psi\psi_y^*)]\,. \quad (3.36b)$$

Example 3.2.9.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + \beta a(u_1 - u_2)u_{1x} + av/\Delta \quad (3.37a)$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + \beta a(u_1 - u_2)u_{2x} - av/\Delta \quad (3.37b)$$

$$v = u_{1x}(2u_{2x} - u_{1y} - u_{2y}) + u_{2x}(u_{1y} + u_{2y}) \quad (3.37c)$$

$$\Delta = 1 - a(u_{1x} + u_{2x})\,. \quad (3.37d)$$

It comes from (2.14) with (3.3) and

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \quad (3.38a)$$

$$H_1(\boldsymbol{u}) = a(u_1 + u_2) \qquad H_2(\boldsymbol{u}) = 0\,. \quad (3.38b)$$

Example 3.2.10.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + a\beta u_1 u_{1x} - b\beta u_2 u_{1y} + a\{2u_{1x}u_{2x} - u_{1x}u_{1y} + u_{2x}u_{2y} + [(2b-a)u_{1x} + bu_{2y}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/\Delta \tag{3.39a}$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + a\beta u_1 u_{2x} - b\beta u_2 u_{2y} + [-2bu_{1x}u_{2y} + b(u_{1y}^2 - u_{2y}^2) + a(1 - au_{2x} + bu_{1y})(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \tag{3.39b}$$

$$\Delta = 1 - au_{2x} - bu_{1y} - ab(u_{1x}u_{2y} - u_{2x}u_{1y}) \,. \tag{3.39c}$$

It comes from (2.14) with (3.38*a*), (3.3) and (3.25). If $a = 0$, equation (3.39) reads

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} - \beta b u_2 u_{1y} \tag{3.40a}$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} - \beta b u_2 u_{2y} + b(u_{1y}^2 - u_{2y}^2 - 2u_{1x}u_{2y})/(1 - bu_{1y}) \,. \tag{3.40b}$$

This pair of equations reproduces (with $u = u_1$) equation (3.8), with the variables x and y exchanged and a replaced by b.

If $b = 0$, equation (3.39) reads

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + a(\beta u_1 - u_{1y})u_{1x} + a(2u_{1x}u_{2x} + u_{2x}u_{2y} - au_{1x}^2 u_{2y})/(1 - au_{2x}) \tag{3.41a}$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + a(\beta u_1 u_{2x} + u_{1x}u_{2y} - u_{2x}u_{1y}) \,. \tag{3.41b}$$

This pair of equations reproduces (with $u = u_2$) equation (3.8) again.

Example 3.2.11.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + \beta(bu_1 - au_2)u_{1y} + b[2u_{2x}u_{1y} + u_{2y}^2 - u_{1y}^2 + 2au_{1y}(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \tag{3.42a}$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + \beta(bu_1 - au_2)u_{2y} + a[2u_{1x}u_{2y} + u_{1y}^2 - u_{2y}^2 + 2bu_{2y}(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \tag{3.42b}$$

$$\Delta = 1 - au_{1y} - bu_{2y} \,. \tag{3.42c}$$

It comes from (2.14) with (3.38*a*), (3.3) and

$$H_1(\boldsymbol{u}) = 0 \qquad H_2(\boldsymbol{u}) = au_1 + bu_2 \,. \tag{3.43}$$

Example 3.2.12.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + \beta a(u_1^2 - u_2^2)u_{1x} + a\{u_1(2u_{1x}u_{2x} - u_{1x}u_{1y} + u_{2x}u_{2y}) - [u_2 + a(u_1^2 - u_2^2)u_{1x}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/[1 - a(u_1u_2)_x] \tag{3.44a}$$

$$u_{2t} = -\beta u_1 + u_{2x} + u_{1y} + \beta a(u_1^2 - u_2^2)u_{2x} + a\{u_2(-2u_{1x}u_{2x} + u_{1x}u_{1y} - u_{2x}u_{2y}) + [u_1 - a(u_1^2 - u_2^2)u_{2x}](u_{1x}u_{2y} - u_{2x}u_{1y})\}/[1 - a(u_1u_2)_x] \,. \tag{3.44b}$$

It comes from (2.14) with (3.38*a*), (3.3) and

$$H_1(\boldsymbol{u}) = au_1u_2 \qquad H_2(\boldsymbol{u}) = 0 \,. \tag{3.45}$$

Example 3.2.13.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + \beta a v_1 v_2 D_1 - a v_1[v_{1y}(v_{2x} + v_{2y}) - 2D_1(u_{2x} + a v_1 D)]/\Delta \quad (3.46a)$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + \beta a v_1 v_2 D_2 + a v_1[v_{1y}(v_{2x} + v_{2y}) - 2D_2(u_{1x} - a v_1 D)]/\Delta \quad (3.46b)$$

$$\Delta = 1 - a v_1(D_1 + D_2) \quad (3.46c)$$

$$v_1 = u_1 + u_2 \quad (3.46d)$$

$$v_2 = u_1 - u_2 \quad (3.46e)$$

$$D_1 = u_{1x} + u_{1y} \quad (3.46f)$$

$$D_2 = u_{2x} + u_{2y} \quad (3.46g)$$

$$D = u_{1x}u_{2y} - u_{2x}u_{1y}\,. \quad (3.46h)$$

It comes from (2.14) with (3.38*a*), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}a(u_1 + u_2)^2 \qquad H_2(\boldsymbol{u}) = \tfrac{1}{2}a(u_1 + u_2)^2\,. \quad (3.47)$$

Example 3.2.14.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} - \beta u_1 u_2(a u_{1x} - b u_{1y}) + [b u_2(2u_{2x}u_{1y} - u_{1y}^2 + u_{2y}^2) - a u_1(1 - a u_1 u_{1x} + b u_2 u_{2y})(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \quad (3.48a)$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} - \beta u_1 u_2(a u_{2x} - b u_{2y}) + [a u_1(-2u_{1x}u_{2x} + u_{1x}u_{1y} - u_{2x}u_{2y}) + a u_1(a u_1 u_{2x} + 2b u_2 u_{2x} - b u_2 u_{1y})(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \quad (3.48b)$$

$$\Delta = 1 - a u_1 u_{1x} - b u_2 u_{2y} + a b u_1 u_2(u_{1x}u_{2y} - u_{2x}u_{1y})\,. \quad (3.48c)$$

It comes from (2.14) with (3.38*a*), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}a u_1^2 \qquad H_2(\boldsymbol{u}) = \tfrac{1}{2}b u_2^2\,. \quad (3.49)$$

If $b = 0$, equation (3.48) reads

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} - \beta a u_1 u_2 u_{1x} - a u_1(u_{1x}u_{2y} - u_{2x}u_{1y}) \quad (3.50a)$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} - \beta a u_1 u_2 u_{2x} - a u_1 u_{2x}u_{2y} + a u_1(u_{1x}u_{1y} - 2u_{1x}u_{2x} - a u_1 u_{2x}^2 u_{1y})/(1 - a u_1 u_{1x})\,. \quad (3.50b)$$

This pair of equations reproduces (3.9) (with $u = u_1$).

If $a = 0$, equation (3.48) reads

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + \beta b u_1 u_2 u_{1y} + b u_2(2u_{2x}u_{1y} - u_{1y}^2 + u_{2y}^2)/(1 - b u_2 u_{2y}) \quad (3.51a)$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + \beta b u_1 u_2 u_{2y}\,. \quad (3.51b)$$

This pair of equations again reproduces (3.9) (but with $u = u_2$, and with the variables x and y exchanged and a replaced by b).

Example 3.2.15.

$$u_{1t} = \beta u_2 + u_{1x} + u_{2y} + au_2[2u_{2x}u_{1y} - u_{1y}^2 + u_{2y}^2 + 2au_1u_{1y}(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \qquad (3.52a)$$

$$u_{2t} = -\beta u_1 - u_{2x} + u_{1y} + au_1[-2u_{1x}u_{2y} + u_{1y}^2 - u_{2y}^2 + 2au_2u_{2y}(u_{1x}u_{2y} - u_{2x}u_{1y})]/\Delta \qquad (3.52b)$$

$$\Delta = 1 - au_1u_{1y} - au_2u_{2y}\,. \qquad (3.52c)$$

It comes from (2.14) with (3.38*a*), (3.3) and

$$H_1(\boldsymbol{u}) = 0 \qquad H_2(\boldsymbol{u}) = \tfrac{1}{2}a(u_1^2 + u_2^2)\,. \qquad (3.53)$$

Example 3.2.16.

$$u_{1t} = u_{2x} \qquad (3.54a)$$

$$u_{2t} = u_{1y} + a(u_{1x}u_{1y} - u_{2x}^2)/(1 - au_{1x}) = (u_{1y} - au_{2x}^2)/(1 - au_{1x})\,. \qquad (3.54b)$$

It comes from (2.14) with (3.3), (3.20) and

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{B} = 0\,. \qquad (3.55)$$

This pair of equations yields (with $u = u_1$)

$$u_{tt} = \left[u_y + a\left(\frac{u_xu_y - u_t^2}{1 - au_x}\right)\right]_x \qquad (3.56a)$$

or equivalently

$$u_{tt} = (u_{xy} - 2au_tu_{tx})/(1 - au_x) + a(u_y - au_t^2)u_{xx}/(1 - au_x)^2\,. \qquad (3.56b)$$

Example 3.2.17.

$$u_{1t} = u_{2x} \qquad (3.57a)$$

$$u_{2t} = u_{1y} + au_1(u_{1x}u_{1y} - u_{2x}^2)/(1 - au_1u_{1x}) = (u_{1y} - au_1u_{2x}^2)/(1 - au_1u_{1x})\,. \qquad (3.57b)$$

It comes from (2.14) with (3.55), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}au_1^2 \qquad H_2(\boldsymbol{u}) = 0\,. \qquad (3.58)$$

This pair of equations yields (with $u = u_1$)

$$u_{tt} = \left[u_y + au\left(\frac{u_xu_y - u_t^2}{1 - auu_x}\right)\right]_x \qquad (3.59a)$$

or equivalently

$$u_{tt} = (u_{xy} - 2auu_tu_{tx})/(1 - auu_x) + a[u_x^2u_y - u_xu_t^2 + u(u_y - auu_t^2)u_{xx}]/(1 - auu_x)^2\,. \qquad (3.59b)$$

Example 3.2.18.

$$u_{1t} = u_{2x} + a(u_{1x}u_{2y} - u_{2x}u_{1y}) \tag{3.60a}$$

$$u_{2t} = u_{1y} - au_{2x}u_{2y} + a(u_{1y}^2 - au_{1x}u_{2y}^2)/(1 - au_{1y})$$
$$= -au_{2x}u_{2y} + (u_{1y} - a^2u_{1x}u_{2y}^2)/(1 - au_{1y})\,. \tag{3.60b}$$

It comes from (2.14) with (3.55), (3.3) and

$$H_1(\boldsymbol{u}) = 0 \qquad H_2(\boldsymbol{u}) = au_1\,. \tag{3.61}$$

This pair of equations reproduces (with $u = u_1$) (3.57) with the variables x and y exchanged.

Example 3.2.19.

$$u_{1t} = u_{2x} - au_2u_{1x}u_{1y} + au_2(u_{2x}^2 - au_2u_{1x}^2u_{2y})/(1 - au_2u_{2x}) \tag{3.62a}$$

$$u_{2t} = u_{1y} + au_2(u_{1x}u_{2y} - u_{2x}u_{1y})\,. \tag{3.62b}$$

It comes from (2.14) with (3.55), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}au_2^2 \qquad H_2(\boldsymbol{u}) = 0\,. \tag{3.63}$$

This pair of equations reproduces (with $u = u_2$) precisely (3.59).

Example 3.2.20.

$$u_{1t} = u_{2x} + abv \tag{3.64a}$$

$$u_{2t} = u_{1y} - av \tag{3.64b}$$

$$v = [u_{2x}^2 - u_{1x}u_{1y} - b(u_{1x}u_{2y} - u_{2x}u_{1y}) + b^2(u_{1y}^2 - u_{2x}u_{2y})]/\Delta \tag{3.64c}$$

$$\Delta = 1 - a(u_{1x} + bu_{2x} - b^2u_{1y} - b^3u_{2y})\,. \tag{3.64d}$$

It comes from (2.14) with (3.55), (3.3) and

$$H_1(\boldsymbol{u}) = a(u_1 + bu_2) \qquad H_2(\boldsymbol{u}) = -b^2a(u_1 + bu_2)\,. \tag{3.65}$$

Example 3.2.21.

$$u_{1t} = u_{2x} + v \tag{3.66a}$$

$$u_{2t} = u_{1y} - v \tag{3.66b}$$

$$v = a(u_1 + u_2)[-u_{1x}(u_{1y} + u_{2y}) + u_{2x}(u_{2x} + u_{1y} - u_{2y}) + u_{1y}^2]/\Delta \tag{3.66c}$$

$$\Delta = 1 - a(u_1 + u_2)(u_{1x} + u_{2x} - u_{1y} - u_{2y})\,. \tag{3.66d}$$

It comes from (2.14) with (3.55), (3.3) and

$$H_1(\boldsymbol{u}) = \tfrac{1}{2}a(u_1 + u_2)^2 \qquad H_2(\boldsymbol{u}) = -\tfrac{1}{2}a(u_1 + u_2)^2\,. \tag{3.67}$$

Example 3.2.22.

$$u_{1t} = u_{2x} + au_1[u_{2x}^2 - u_{1x}u_{1y} - au_1u_{1x}(u_{1x}u_{2y} - u_{2x}u_{1y})]/[1 - a(u_1u_2)_x] \tag{3.68a}$$

$$u_{2t} = u_{1y} + a[u_2(u_{1x}u_{1y} - u_{2x}^2) + u_1(1 - au_1u_{2x})(u_{1x}u_{2y} - u_{2x}u_{1y})]/[1 - a(u_1u_2)_x]\,. \tag{3.68b}$$

It comes from (2.14) with (3.55), (3.3) and (3.45).

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References

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